Exam Modeling and Control of Complex Nonlinear Engineering Systems

April 10, 2014, 14:00 – 17:00 hours

Note: Version for **Mathematics** students

- Make sure that you **MOTIVATE** your answers!
- Don't forget to put your name and id on every page you hand in.
- The exam is '*open-book*': you may use the Lecture Notes and all written material.
- The final grade is determined by 60 % of this exam, and 40 % of the average of the homeworks.
- Good luck!

Problem 1

(a) Consider the system

$$
\begin{aligned}
\dot{q}_1 &= u_1 \\
\dot{q}_2 &= u_2 \\
\dot{q}_3 &= q_1 u_2 - q_2 u_1 \\
\dot{q}_4 &= q_1^2 u_2 \\
\dot{q}_5 &= q_2^2 u_1 \\
y &= q_3
\end{aligned}
$$

Show that the system is controllable, while its linearization around any point $\bar{q}_1, \bar{q}_2, \bar{q}_3, \bar{q}_4, \bar{q}_5$, $\bar{u}_1 = 0, \bar{u}_2 = 0$ is uncontrollable.

(b) Show that the system is not locally observable.

Problem 2

(a) Consider a system

$$
\begin{aligned}
\dot{x} &= f(x) + g(x)u, \quad x \in \mathbb{R}^n, u, y \in \mathbb{R} \\
y &= h(x)
\end{aligned}
$$

which is passive, with a storage function $S(x) \geq 0$ whose $n \times n$ Hessian matrix satisfies

$$
\frac{\partial^2 S}{\partial x^2}(x) > 0, \quad \text{for all } x.
$$

Prove that the system satisfies $L_g h(x) \neq 0$ for all x, and therefore is everywhere input-output linearizable. What is the dimension of its zero-dynamics ?

(b) Consider the system

$$
\begin{aligned}
\dot{x}_1 &= -x_1 + k(x_2) \\
\dot{x}_2 &= -x_1 - 2k(x_2) + u \\
y &= x_2
\end{aligned}
\tag{1}
$$

where $k : \mathbb{R} \to \mathbb{R}$, $k(0) = 0$ and $k'(x_2) > 0$ for all x_2. Show by using the candidate storage function

$$
S(x_1, x_2) = \frac{1}{2} x_1^2 + \int_0^{x_1} k(z) dz
$$

that the system is strictly output passive, and that $x_1 = 0, x_2 = 0$ is an asymptotically stable equilibrium for $u = 0$.

(c) Consider again the system (1). Suppose one wants to asymptotically track, for an arbitrary initial condition $x(0)$, a given reference trajectory

$$y_{\text{ref}}(t) = \arctan t, \quad t \geq 0$$

in the sense that $\lim_{t\to\infty} |y(t) - y_{\text{ref}}(t)| = 0$.
Show how to do this using input-output linearization.

Problem 3

(a) Consider a nonlinear control system

$$\Sigma : \dot{x} = g_1(x)u_1 + g_2(x)u_2, \quad x \in \mathbb{R}^n$$

with two inputs u_1, u_2.
Now suppose that u_1, u_2 are not the actual controls but instead their derivatives $v_1 = \dot{u}_1, v_2 = \dot{u}_2$. This leads to the definition of the *extended* system

$$\Sigma_e : \begin{aligned} \dot{x} &= g_1(x)u_1 + g_2(x)u_2, \quad x \in \mathbb{R}^n \\ \dot{u}_1 &= v_1 \\ \dot{u}_2 &= v_2 \end{aligned}$$

with to inputs v_1, v_2, and *extended* state (x, u_1, u_2).
Prove that Σ is controllable if and only if the extended system Σ_e satisfies the strong accessibility rank condition at every point (x, u_1, u_2).

(b) Try to give arguments why actually the following stronger property holds: Σ is controllable if and only if the extended system Σ_e is controllable.
Furthermore, argue that the 'bad brackets' (see Section 3.5.3) do not play a role for the controllability analysis of Σ_e.

Problem 4

Consider the following system

$$\dot{x} = \begin{bmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{bmatrix} x, \quad x \in \mathbb{R}^3,$$

which describes the dynamics of a rigid body rotating around its center of mass subject to angular velocities u_1, u_2, u_3 around its principal axes (which are assumed to be the control inputs).

(a) Write this system into the form

$$\dot{x} = u_1 B_1 x + u_2 B_2 x + u_3 B_3 x$$

for the rotation matrices

$$B_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, B_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Prove that the system is *not* controllable from any point $x_0 \in \mathbb{R}^3$.

(b) Prove that the reachable set from any point x_0 is equal to the sphere

$$B_{x_0} = \{x \in \mathbb{R}^3 \mid \|x\| = \|x_0\|\}$$

(c) Prove that the controllability properties of the system are not changed if we take one of the three control inputs u_1, u_2, u_3 equal to zero. What is happening if we take two inputs to be zero ?

Problem 5

Consider a simple voltage-driven RC circuit depicted in Figure 1. Assume that $R_1 > 0$ is a linear resistor, and $C > 0$ a linear capacitor. Furthermore, assume that R_2 is a nonlinear resistor with characteristic $R_2(v_c) = v_c + v_c^3$.

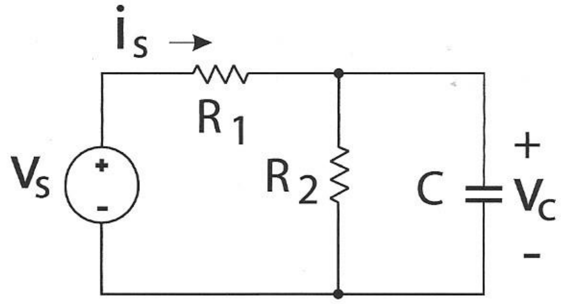

Figure 1: Simple voltage-driven RC circuit.

The dynamical equations are described by

$$C\dot{v}_c = \frac{1}{R_1}v_S - \frac{1}{R_1}v_c - v_c - v_c^3$$

a) Show that the input v_S/R_1 and the output v_C is a passive input output pair.

Consider the following function:

$$G(v_C) = \frac{1}{2R_1}v_C^2 + \frac{1}{2}v_C^2 + \frac{1}{4}v_C^4.$$

b) Show that $(v_S/R_1, \dot{v}_C)$ is a passive input-output pair).

c) Suppose that $v_S = 0$. Is $G(v_C)$ a possible candidate Lyapunov function? Is the system stable in the sense of Lyapunov's direct method? Motivate your answer!